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ACCEPTANCE SAMPLING OF DISCRETE CONTINUOUS PROCESSES

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In the paper we present an overview of statistical procedures that have been proposed for the inspection of discrete continuous processes. The overview covers single level CSP-type and WSP-type sampling plans, multi-level sampling plans, and other continuous sampling plans of different types. We also present proposals for future standardized continuous sampling plans.

Keywords: inspection, acceptance sampling, sampling plans

1. Introduction

Procedures of statistical quality control are traditionally attributed to two main areas: acceptance sampling and statistical process control. The main aim of the oldest procedures of acceptance sampling, known as *acceptance sampling plans*, is to inspect certain items (products, documents, etc.) submitted for inspection in lots or batches. First acceptance sampling plans, proposed by one of the fathers of SQC, Harold Dodge, were designed for the inspection of lots submitted in sequences (lot-by-lot inspection). The only aim of those plans, known as Dodge-Romig LTPD plans or Dodge-Romig AOQL plans, was to "screen" the inspected series of lots, and to reject lots of supposedly "spotty" quality. In case of stable production processes, i.e. processes characterized by constant probabilities of producing nonconforming items, high quality requirements can be achieved by occasional screening of rejected lots.

The aim of the second main area of SQC, statistical process control (SPC), introduced by Walter Shewhart, was different. Statistical procedures of SPC are used for monitoring processes with the aim of triggering alarms if they deteriorate. Because of the different aims, the procedures of acceptance sampling have been called "passive" in contrast to "active" procedures of SPC. It has to be stressed, however, that both of these "labels" are somewhat misleading. Procedures of SPC do not indicate measures which have to be taken in order to improve controlled processes. Thus, their "active" character is somewhat questionable, especially by specialists in automatic process control (APC). On the other hand, the application of acceptance sampling plans does not mean that the results of inspection cannot be used as "active" signals indicating the necessity of process improvements. Critical opinions formulated against traditional acceptance sampling procedures have motivated statisticians to building acceptance sampling schemes and systems with additional “pro-active” features, like those of the sampling systems presented in international standards of ISO 2859 and ISO 3951 series.

The concept of the acceptance sampling of lots submitted for inspection either in series or in isolation is typical of commercial activities of producers and consumers. Therefore, acceptance sampling plans, in contrast to control charts – the most popular tools of SPC – are mainly used for inspection of final products. This raises questions about their usefulness, as the real quality of a product is built-in during a production process. Therefore, one could ask a question about the possibility of using acceptance sampling procedures during the production process. The affirmative answer to this question was given by Harold Dodge (1943), who introduced continuous sampling plans. The idea behind these statistical procedures is exactly the same as in original acceptance sampling plans for lot-by-lot inspection, i.e. to “screen” production processes, but not only at their final stages. The same idea had motivated Wald and Wolfowitz (1945), who, at the same time, proposed other statistical procedures used for screening of continuous production processes.

The original procedure proposed by Dodge (1943) has some undesirable features. Thus, many authors, including Dodge himself, have tried to modify and extend it in order to arrive at procedures with better properties. The results of their efforts are overviewed in the second and the third sections of this paper. In the second section we present the original Dodge’s CSP-1 plan and its different extensions. In the third section we present some multi-level generalizations of the CSP sampling plan. The procedure proposed by Wald and Wolfowitz (1945), named later on WSP-1, and its further extensions are presented in the fourth section of the paper.

Other approaches to the inspection of continuous discrete processes also exist. They are using such statistical techniques as runs and cumulative sums. They are overviewed in the fifth section of the paper. The main focus is on the procedure proposed by Beattie (1962), which seems to be the most interesting one from the point of view of its possible future applications.

Parameters of continuous sampling plans are usually found using a purely statistical approach. However, it is also possible to design such procedures using some economic considerations. Some examples of the economic approach to design continuous sampling plans are sketched in the sixth section of the paper. In the seventh section we briefly present the only existing standard on continuous sampling, namely the MIL-STD-1235C. This standard is obsolete, and a possible new standard on continuous sampling should be based on other statistical procedures.

These new procedures should be regarded as some modifications of the existing continuous sampling procedures. The aim of introducing these modifications should be similar to that behind the SPC procedures like control charts. The modified continuous sampling plans should have, in our opinion, built-in automatic procedures for triggering alarms in the case of deterioration of the inspected process. In the last section of the paper we present some proposals on how to achieve this goal.

Introducing such modifications and extensions should be regarded as a prerequisite for future standardization of the proposed continuous sampling plans.

2. Continuous sampling plans of the CSP-type

2.1 CSP-1 continuous sampling plan

The first type of acceptance sampling plan for attribute sampling from a continuous production process, known as the CSP-1 continuous sampling plan, was proposed by Harold Dodge in his paper Dodge (1943). The aim of this procedure is to *rectify* the inspected process in order to have a low fraction of nonconforming items at its output. This aim is achieved by alternating between 100% inspection (screening) and sampling at a frequency $f=1/n$. The original CSP-1 procedure consists of two steps.

Step 1: At the outset, inspect 100% items taken from a process until i consecutive conforming items are observed. Then, go to Step 2.

Step 2: Discontinue 100% inspection, and inspect only a fraction f of the units until a sample unit is found nonconforming. Then, revert immediately to 100% inspection, i.e. to Step 1.

The sampling method should assure an unbiased sample. Three methods that fulfill this requirement are available:

a) sampling each item with probability $f=1/n$ (probability sampling),
b) sampling every nth item (systematic sampling),
c) sampling one item taken randomly from every segment of n items (random sampling).

Nonconforming items found during the inspection can be either removed from the process or replaced by conforming ones. All continuous sampling procedures that can be described by such two steps (consisting of other possible sub-steps) are called CSP-type continuous sampling plans.

The basic statistical characteristic of all CSP-type continuous sampling plans is the Average Fraction Inspected (*AFI*), considered as a function of fraction of nonconforming p, which in case of the CSP-1 sampling plan defined by Dodge is

$$F(p)=\frac{u+fv}{u+v}, \tag{1}$$

where u is the expected duration of Step 1, and v is the expected duration of Step 2. In a general case of the CSP-type sampling plans fv in (1) should be replaced by the expected number of items inspected during the Step 2 of the procedure. For the CSP-1 sampling plan the formulae for u and v have been derived by Dodge (1943) under the assumption that consecutive items are described by independent and identically distributed (*iid*) Bernoulli random variables. Under this assumption we have

$$u=\frac{1-(1-p)^i}{p(1-p)^i}=\frac{1-q^i}{pq^i} \tag{2}$$

and

$$v = \frac{1}{fp}. \tag{3}$$

Hence,

$$F(p) = \frac{f}{f + (1-f)(1-p)^i}. \tag{4}$$

When nonconforming items are simply removed from the process, the *clearance number i* in (4) should be replaced with *i*-1. In this case the *AFI* is computed in relation to the output of the process, in contrast to the original case in which the nonconforming items are replaced with conforming ones, when it is computed in relation to the outset of the process.

The second important characteristic of the acceptance sampling plan is the Average Outgoing Quality (*AOQ*) defined as

$$AOQ(p) = p[1 - F(p)]. \tag{5}$$

In case of the CSP-1 sampling plan we have

$$p_A = AOQ(p) = p\left[1 - \frac{f}{f + (1-f)(1-p)^i}\right]. \tag{6}$$

As the value of *p* may not be known in advance, Dodge (1943) proposed to describe the plan by the characteristic introduced previously by himself in the context of acceptance sampling of lots, namely the Average Outgoing Quality Limit (*AOQL*), defined as

$$AOQL = p_L = \max_p \{p[1 - F(p)]\}. \tag{7}$$

For the CSP-1 sampling plan the *AOQL* cannot be expressed in a closed form as a function of parameters *i* and *f*. However, if we assume that *AOQ*(*p*) attains its maximum equal to p_L when the fraction nonconforming is equal to p_1 we have the following relation (Dodge, 1943) linking both parameters of the plan

$$f = \frac{(1-p_1)^{i+1}}{ip_L + (1-p_1)^{i+1}}. \tag{8}$$

For given values of the clearance number *i* and the AQQL equal to p_L we can find the value of p_1 from the equation

$$p_1 = \frac{1 + ip_L}{i+1}. \tag{9}$$

Hence, we can insert (9) into (8), and obtain the relation between *i* and *f*.

Now, the problem how to design the CSP-1 sampling plan that fulfills the requirement on AOQL boils down to the setting of a second requirement. Dodge (1943) set a limit for the probability α of not-detecting a nonconforming item during the sampling inspection of N consecutive items when the fraction nonconforming has jumped to an unacceptable value p_r. This requirement has the following form

$$\left(1 - fp_r\right)^N \leq \alpha \ , \tag{10}$$

and can be used for the calculation of *f*. Then, one has to calculate the value of *i* from the relationship between *i* and *f* described above.

Another interesting method has been proposed by the Russian statisticians Shor and Pakhomov (1973). They introduced, additionally to the requirement for p_L=AOQL, the following requirement for the fraction of inspected items

$$F\left(p_L\right) \leq \alpha \ . \tag{11}$$

The CSP-1 sampling plan that fulfills these both requirements has the parameters given by the following equations (Shor and Pakhomov, 1973):

$$i = \frac{\left(1/\alpha - 1\right)\left(1 - p_L\right)}{ep_L}, e = 2{,}71.., \tag{12}$$

$$\frac{1}{f} - 1 = \left(\frac{1}{\alpha} - 1\right)\left(1 - p_L\right)^i \ . \tag{13}$$

The formula (12) is valid when nonconforming items are replaced with conforming ones. When nonconforming items found during the inspection are only removed, the formula for the clearance number *i* is given by

$$i = \frac{1/\alpha - 1}{ep_L}, \ e = 2{,}71... \tag{14}$$

Some other criteria for the design of the CSP-1 continuous sampling plans are overviewed in the paper by Phillips (1969).

In the original paper by Dodge (1943) it is assumed that the process fraction nonconforming is constant in time. This assumption was relaxed by Lieberman (1953) who assumed that the inspected process is not under statistical control, and its consecutive items are not described by independent and identically distributed random variables. When probability sampling is used, and nonconforming items found during the inspection are replaced with the conforming ones, the maximal value of the fraction nonconforming at the output of the process, called *Unrestricted Average Outgoing Quality Limit* UAOQL, is given by

$$UAOQL = \frac{1-f}{1+if}. \tag{15}$$

When random sampling is used, the formula for UAOQL, according to Derman et al. (1959), is given by the same formula. The case when nonconforming items are simply rejected was considered by Endres (1969), who showed that the formula for UAOQL in this case is the same as (15), but with *i* replaced by *i*-1.

UAOQL is often criticized as the characteristic which describes properties obtained under hardly realistic conditions. White (1965) has shown that UAOQL describes the average quality at the output of the process controlled by an omniscient "evil demon" who tries to outsmart the inspector. From a mathematical point of view it means that the qualities of consecutively inspected items are not independent, and depend upon the stage of inspection. A much more realistic situation is described by the model introduced by Hillier (1964), who assumed that the process usually operates at an acceptable level p_0 and then suddenly jumps to an unacceptable level p_1. Let D be the number of not inspected nonconforming items among the next L items after the Mth item is observed. He introduced a new criterion, the AEDL (Average Extra Defectives Limit), which is the smallest number such that

$$E(D) \le AEDL + L \cdot AOQL, \tag{16}$$

for all possible values of L, M, p_0 and p_1. The value of AEDL gives us additional information about possible consequences of the process deterioration. For the CSP-1 sampling plan the formula for AEDL can be found in Hillier (1964),

$$AEDL = \begin{Bmatrix} \frac{1-f}{f}\left[1-(1-f)^x\right] - xp_L, & x > 0 \\ 0, & x \le 0 \end{Bmatrix}, \tag{17}$$

where

$$x = \frac{ln\frac{-fp_L}{(1-f)ln(1-f)}}{ln(1-f)}. \tag{18}$$

Properties of the CSP-1 plan are calculated under the assumption of an infinite production run. However, in practice, production runs are of finite length, say N. In such case characteristics of the plan can be computed using the Markov chain approach. This approach has been used by many authors, who investigated the properties of different continuous sampling plans. Some interesting analytical approximate results were presented in papers by Blackwell (1977) and McShane and Turnbull (1991). The results presented in McShane and Turnbull (1991) are more general, as they are also applicable for the case of dependent consecutive inspection results. Yang (1983) applied another approach, namely the theory of renewal processes, and also obtained some useful approximations. For example, in the case of probability sampling she proved that approximately

$$AOQ_N = AOQ_\infty + \frac{EZ}{2N}\left[\frac{\sigma_W^2 + EW}{(EW)^2} - 1\right]. \tag{19}$$

AOQ_∞ is the Average Outgoing Quality for the original CSP-1 plan given by

$$AOQ_\infty = \frac{EZ}{EW}, \tag{20}$$

where

$$EZ = n - 1, \tag{21}$$

$$EW = \frac{1 + (n-1)q^i}{pq^i}, \tag{22}$$

$$\sigma_W^2 = \sigma_\tau^2 + \frac{n}{p}\left(\frac{n}{p} - 1\right), \tag{23}$$

$$\sigma_\tau^2 = \left[1 - pq^i(2i+1) - q^{2i+1}\right] / \left[p^2 q^{2i}\right], \tag{24}$$

with n=1/f and q=1-p. Similar, but slightly different in case of σ_W^2, formulae have been derived by Yang (1983) for the case of random and stratified sampling.

2.2 Modifications of the CSP-1 continuous sampling plan

The weakest point of the CSP-1 continuous sampling plan is its rule for switching from sampling to screening. Inspection has to be switched to its screening phase immediately after only one nonconforming item has been found during the sampling phase. This creates significant problems related to frequent changes of the intensity of inspection, and thus, to important organizational problems. Dodge and Torrey (1951) proposed first modification of the CSP-1 plan, designated CSP-2. In the CSP-2 plan, when a nonconforming item is found, inspection is continued at the same fraction f, and 100 per cent screening is reverted to only if another nonconforming item is found within the next k items. Usually k is taken to be equal i. The AOQ function of this plan is given by the following expression

$$AOQ(p) = p\frac{(1-f)(1-p)^i\left[2-(1-p)^k\right]}{f\left[1-(1-p)^i\right]\left[1-(1-p)^k\right] + (1-p)^i\left[2-(1-p)^k\right]}. \tag{25}$$

This plan tolerates accidental nonconforming items, but does not provide sufficient protection against sudden worsening of the inspected process. Therefore, Dodge and Torrey (1951) proposed its modification, known as CSP-3 plan. This plan specifies inspecting the next four consecutive items after observing a nonconforming item during the sampling phase of inspection. If one of these items is nonconforming the inspection reverts immediately to its screening phase. Otherwise, the sampling phase is continued according to the rules of the CSP-2 sampling plan. The AOQ function of this plan is the following

$$AOQ(p) = p\frac{(1-f)q^i\left[1+q^4-q^{k+4}\right]}{f\left(1-q^i\right)\left(1-q^{k+4}\right)+q^i\left[1+q^4-q^{k+4}\right]+4fpq^i}, \tag{26}$$

where q=1-p. The more general formula, with the arbitrary length of the 100 per cent inspection sequence during the sampling phase, can be found in Yang (1983).

Modifications of the CSP-1 plan proposed in Derman et al. (1959) are valid when random sampling is used during the sampling phase. In the CSP-4 sampling plan, when an item randomly chosen for inspection from a segment of k=1/f items is found nonconforming, the whole of this segment is rejected, and the inspection reverts to its screening phase. The AOQ function in this case is given by

$$AOQ(p) = \frac{(k-1)p(1-p)^{i+1}}{1+(k-1)p(1-p)^{i+1}}. \tag{27}$$

Derman et al. (1959) obtained also the following formula for AOQL

$$AOQL = 1 + q_{m,4}\frac{i+2}{i+1}, \tag{28}$$

where $q_{m,4}$ is the solution of the following equation

$$(k-1)q^{i+2} + (i+2)q = i+1. \tag{29}$$

Formula for UAOQL for the CSP-4 is given in Derman et al. (1959)

$$UAOQL = \begin{cases} \dfrac{(c_4+2)-2\sqrt{c_4+1}}{c_4^2}, & c_4 \neq 0 \\ 0{,}25 & c_4 = 0 \end{cases}, \tag{30}$$

where

$$c_4 = \frac{i-k+1}{k}. \tag{31}$$

In the CSP-5 plan proposed in Derman et al. (1959) the segment with a nonconforming item is screened, and the inspection reverts to its screening phase. The AOQ function in this case is given by

$$AOQ(p) = \frac{(k-1)p(1-p)^{i+1}}{1+(k-1)p(1-p)^{i}}. \tag{32}$$

The AOQL for this plan can be computed from the formula (Derman et al., 1959)

$$AOQL = \frac{(i+1)q_{m,5} - (i+2)q_{m,5}^2}{i}, \tag{33}$$

where $q_{m,5}$ is the solution of the following equation

$$2(k-1)q^{i+1} - (k-1)q^i + (i+2)q = i+1. \tag{34}$$

Formula for the UAOQL for the CSP-5 is given in Derman et al. (1959) in the following form

$$UAOQL = \begin{cases} \dfrac{(c_5+2) - 2\sqrt{c_5+1}}{c_5^2}, & c_5 \neq 0 \\ 0{,}25 & c_5 = 0 \end{cases}, \tag{35}$$

where

$$c_5 = \frac{i}{k}. \tag{36}$$

Some interesting modifications of the CSP-1 sampling plan result from relaxing the rules for switching from sampling to screening. Govindaraju and Kandasamy (2000) proposed a new plan, designated CSP-C, whose sampling inspection phase is terminated when the total number of found nonconforming items found exceeds a certain constant *c*. This continuous sampling plan has been further generalized by Balamurali et al. (2005) who proposed a plan designated CSP-(C_1,C_2).

The sampling phase of the CSP-(C_1,C_2) plan is ruled by the following algorithm:

a) When a screening phase is terminated (according to the rules of the CSP-1 plan), units are inspected at a rate f_1, and the number of nonconforming items found d is counted;
b) When d exceeds a first critical number c_1, sampling inspection is continued, but at a higher rate $f_2 \geq f_1$, and the counting of nonconforming sampled units is continued;
c) When d exceeds a second critical number c_2, sampling inspection is terminated, and inspection reverts to the screening phase.

All found nonconforming items found are corrected or replaced with conforming ones.

The main characteristics of the CSP-(C_1,C_2) sampling plan have been calculated in Balamurali et al. (2005) using the Markov-chain approach. The average number inspected is given by the same formula as in the case of the CSP-1 plan, i.e. by (2), and the average fraction inspected in the long run is given by

$$F(p) = \frac{f_1 f_2\left(1 + c_2 q^i\right)}{f_1 f_2\left(1 - q^i\right) + f_2\left(c_1 + 1\right)q^i + f_1\left(c_2 - c_1\right)q^i}\,. \tag{37}$$

It is easy to notice that in case of $f_1 = f_2 = f$ and $c_1 = c_2 = 0$ the CSP-(C_1,C_2) plan is reduced to the CSP-1 plan. When $c_1 = c_2 = c$, this plan reduces to the CSP-C plan by Govindaraju and Kandasamy (2000).

In all modifications of the CSP-1, mentioned above, the sampling phase is changed in comparison to the original Dodge's solution. Belyaev (1975) proposed an interesting modification of the decision rule for the screening phase. In his continuous sampling plan, designated as critical continuous sampling plan, the decision algorithm for the screening phase is the following:

a) When the first $l=i$ inspected items are conforming begin sampling inspection according to the rules of the CSP-1 ;
b) When the k-th ($k<i$) inspected item is found non-conforming, start the screening phase anew, but with a larger clearance number equal to $l\text{-}k\text{+}i$;
c) When the number of consecutively inspected items k is equal to the current value of the clearance number l stop screening, and switch to the sampling phase.

The average outgoing quality function AOQ(p) for this sampling plan is given by the formula

$$AOQ(p) = (1 - f)\frac{p(1 - ip)}{1 - (1 - f)ip},\ ip \le 1\,. \tag{38}$$

It is interesting to note that for i such that $ip \ge 1$ the average outgoing fraction nonconforming in the long run tends to zero. It means that in the long run the sampling process will remain with probability one in the screening phase.

The expected number of non-conforming items accepted during the inspection process is given by the following formula (Belyaev, 1975)

$$D(p)\frac{1}{1 - \pi}\left(\frac{1}{f} - 1\right), \tag{39}$$

where π is the solution to the following equation

$$\pi = (1 - p + p\pi)^i\,. \tag{40}$$

Parameters of Belyaev's plan are calculated using the condition on the fraction inspected (for a given value of p), together with the minimization of $D(p)$.

3. Multi-level continuous sampling plans

One of disadvantages of Dodge's CSP-1 sampling plan is its high inspection rate during the sampling phase, which is unnecessary in the case of good quality of inspected items. Lieberman and Solomon (1955) introduced a multi-level continuous sampling plan, designated as MLP, in which the sampling rate is decreased when the history of inspection shows good quality of previously inspected items. The MLP plan has k levels of sampling, and in its general case is described by the set of parameters $(i_0, i_1, \ldots, i_k, f_1, \ldots f_k)$. It operates according to the following general algorithm:

Step 0) At the outset, inspect 100% items taken from a process until i_0 consecutive conforming items are observed. Then, go to Step 1.

Step 1) Discontinue 100% inspection and inspect only a fraction f_1 units. If the next i_1 units are conforming, proceed to the next level (Step 2); if a nonconforming item occurs, revert immediately to 100% inspection (Step 0).

Step 2) Discontinue sampling at rate f_1 and proceed to sampling at rate f_2. If the next i_2 units are conforming, proceed to the next level (Step 3); if a nonconforming item occurs, revert to the previous inspection level (Step 1).

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Step j) Discontinue sampling at rate f_{j-1} and proceed to sampling at rate f_j. If the next i_j units are conforming, proceed to the next level (Step j+1); if a nonconforming item occurs, revert to the previous inspection level (Step j-1).

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Step k) Discontinue sampling at rate f_{k-1} and proceed to sampling at rate f_k. If a nonconforming item occurs, revert to the previous inspection level (Step k-1); otherwise continue sampling at rate f_k.

When k=1, the MLP plan is reduced to the CSP-1 plan. The $AOQ(p)$ function of the MLP plan was derived using the Markov-chain approach, and given in (Lieberman and Solomon, 1955) by a very complex formula. Usually, we set $i_0 = i_1 = \cdots = i_k = i$, and $f_k = f^j, j = 1, \ldots, k$, and in this special case we have (Lieberman and Solomon, 1955):

$$AOQ(p) = pz\left\{\frac{1 - z^k}{1 - z^{k+1}} - \frac{f(1 - z)}{1 - fz}\frac{1 - (fz)^k}{1 - z^{k+1}}\right\}, \tag{41}$$

where

$$z = \frac{1}{f}\frac{(1 - p)^i}{1 - (1 - p)^i}\ . \tag{42}$$

Parameters of the MLP plan can be found using the concept of constant AOQL contours introduced in Dodge (1943) for the CSP-1 plan. Let p_L=AOQL. Lieberman and Solomon (1955) found the following approximate formula

$$f_k \approx f_\infty \left[1 - \left(\frac{1}{k}\right)^{1/3}\right] + f_1 \left(\frac{1}{k}\right)^{1/3}, \tag{43}$$

where

$$f_1 = \frac{(1-p_L)^i}{(1-p_L)^i + \left(1+\frac{1}{i}\right)^i (1+i)\frac{p_L}{1-p_L}}, \tag{44}$$

and

$$f_\infty = \frac{(1-p_L)^i}{1-(1-p_L)^i} \tag{45}$$

are constant AOQL contours for k=1 and $k \to \infty$, respectively.

Lieberman and Solomon (1955) conjectured that for the MLP plan there exists a certain UAOQL value. The algorithm for finding this characteristic was proposed by White (1965) in the form of a linear programming problem.

The MLP continuous sampling plan was generalized in the paper by Derman et al. (1957). These authors considered three multi-level tightened continuous sampling plans. The plans are called tightened, as they allow reversion to the 100% screening more quickly than the original MLP plan. This feature is very useful when the inspected process deteriorates at some unknown moment.

In the case of the MLP-r×1 sampling plan, a systematic sampling procedure is used. If, at the j-th level of the plan, i consecutive inspected items are found conforming, the sampling inspection switches to the next level characterized by a lower inspection rate. However, if a nonconforming item is found, the inspection process goes back to the (j-r)-th level if $j>r$, or to the 100 % inspection (zero level) otherwise. The MLP plan proposed by Lieberman and Solomon (1955) is obviously the MLP-1×1 sampling plan.

The next plan proposed by Derman et al. (1957) is designated as MLP-T. For this plan the inspection process is always switched to the 100% inspection when a nonconforming item is found during sampling inspection.

The third continuous sampling plan proposed by Derman et al. (1957), designated as MLP-r×s, is the most complicated procedure, generalizing the MLP-r×1 sampling plan. According to this plan, if i consecutive items are found conforming during the sampling phase at the j-th level, the inspection switches to the (j+s)-th level. However, when a non-conforming item is found, the inspection process goes back to the (j-r)-th level if $j>r$, or to the 100 % inspection (zero level) otherwise. It is worth noticing that for r=s the MLP-r×s has the same properties as the MLP sampling plan.

Derman et al. (1957) derived the following formula for the average fraction inspected $F(p)$ for the MLP-T tightened plan

$$\frac{1}{F(p)} = \left(1-q^i\right)\frac{1-\left(q^i/f\right)^k}{1-q^i/f} + \left(q^i/f\right)^k, \; q = 1-p \; . \tag{46}$$

They also found closed-form formulae for the values of AOQL, but only for the case of an infinite number of inspection levels. In the case of the MLP-r×1 sampling plan, the AOQL is given by the following expression

$$AOQL = 1-\left(\frac{f-f^{r+1}}{1-f^{r+1}}\right)^{1/i}, \tag{47}$$

and by

$$AOQL = 1-f^{1/i} \tag{48}$$

in the case of the MLP-T sampling plan. Additional variants of the MLP-T sampling plan were proposed by Guthrie and Johns (1958) who considered alternative sequences of sampling rates.

In the case of an unstable process, the value of AEDL of this plan was found by Hillier (1964) and is given by the following formula

$$AEDL = \begin{cases} \frac{1-f^k}{f^k}\left[1-\left(1-f^k\right)^x\right] - xp_L, & x > 0 \\ 0, & x \le 0 \end{cases}, \tag{49}$$

where

$$x = \frac{ln\left[\frac{-f^k p_L}{\left(1-f^k\right)ln\left(1-f^k\right)}\right]}{ln\left(1-f^k\right)} . \tag{50}$$

Multi-level continuous sampling plans like MLP or MLP-T do not allow an immediate switch to low sampling rates when the history of screening shows good quality of the inspected process. Sackrowitz (1972) proposed alternative multi-level plans that are fully equivalent to the MLP or MLP-T plans (i.e. they have the same statistical characteristics in case of stable quality of the inspected process), but allow a return to a high sampling level more quickly than can be done in the case of the MLP or MLP-T plans. In the plans proposed by Sackrowitz (1972), 100% inspection is switched to sampling when i consecutive conforming items are found, but in contrast to MLP and MLP-T plans, the type of sampling inspection that follows the screening phase depends on the length of time that is needed for switching to the sampling phase.

In case of the sampling plan, designated by Sackrowitz (1972) P^*, which is fully equivalent to the MLP-T sampling plan, the switching rules are the following:

1) If the first i items inspected during the initial screening phase are found conforming, switch to the second level of sampling characterized by the sampling rate f_2. Otherwise, when i consecutive items inspected during the initial screening phase are found conforming, switch to the first level of sampling characterized by the sampling rate $f_1 > f_2$.
2) When on f_i , $i=1,\ldots,m,$ sampling inspection level, continue sampling until a nonconforming item is found. When it occurs, revert immediately to 100% screening. If the first i items inspected during this screening phase are found conforming, switch to the j^*, $j^*=\min(j+1,m)$ level of sampling, characterized by the sampling rate f_{j^*} . Otherwise, when i consecutive items inspected during this screening phase are found conforming, switch to the first level of sampling, characterized by the sampling rate f_1.

In case of the sampling plan designated by Sackrowitz (1972) P^{**}, which is fully equivalent to the MLP sampling plan, the switching rules are rather complicated, and this feature limits the practicability of this plan.

Probably the most general multi-level continuous sampling procedure has been proposed by Sakamoto and Kurano (1978). These authors propose finding optimal values of the parameters of that procedure using a very complicated model of a stochastic game.

4. Continuous sampling plans of the WSP-type

Wald and Wolfowitz (1945) in their seminal paper noticed that Dodge's CSP sampling plan does not assure a prescribed AOQL level when the quality of the inspected process varies in time. In order to avoid this problem, they proposed a sampling procedure, labeled in their paper SPC, which later on has been designated WSP-1. The WSP-1 sampling plan is based on a concept different from the CSP sampling plans, and the plans based on this concept are called the WSP-type sampling plans. In the case of the WSP-type continuous sampling plans the inspected process is divided into groups of N items each. These groups, if needed, may be diverted from the original process for further screening. Therefore, the WSP-type sampling plans may be regarded as rectification procedures for batches of N items each.

In the original WSP-1 sampling plan the group of N items is divided into n segments of $1/f$ items each. For each of the groups, inspection begins by sampling one item taken at random from consecutive segments. This sampling phase is continued until c+1 nonconforming items are found or until all the segments have been sampled. If the $(c+1)^{st}$ nonconforming item is found in the j^{th} segment, the remaining *n-j* segments are diverted to 100% screening, and sampling of the next group of *N* items begins.

Wald and Wolfowitz (1945) proved that the parameter *c* is the smallest integer that fulfills the inequality

$$\frac{(c+1)(1-f)}{n} > AOQL\ . \tag{51}$$

Shahani (1979) investigated other properties of the WSP-1 sampling plan. He showed that the proportion of inspected items, both in sampling and screening phases, is given by the following expression

$$F_1 = 1-(1-f)\left\{P_a(n,c)+\frac{1}{n}\sum_{j=c+1}^{n} jP_j\right\}, \tag{52}$$

where

$$P_a(n,c) = 1-\sum_{j=c+1}^{n} P_j\ , \tag{53}$$

and

$$P_j = \binom{j-1}{c} p^{c+1}(1-p)^{j-1-c},\ c+1 \le j \le n\ . \tag{54}$$

Then, Shahani (1979) also showed that the AOQ function

$$AOQ(p) = p(1-F_1) \tag{55}$$

for the WSP-1 sampling plan is an increasing function, attaining its maximum at $p=1$. This maximum is equal to the value of (unrestricted) AOQL in (51). Therefore, the WSP-1 sampling plan in the most unfavorable condition does not provide protection against bad quality.

Read and Beattie (1961) modified the WSP-1 procedure, and proposed a plan, named later WSP-2. According to the WSP-2 plan, when the $(c+1)^{st}$ nonconforming item is found in the j^{th} segment, all the *j* sampled segments are diverted for screening and the count of a new group of *N* items begins. The proportion of inspected items for the WSP-2 plan is given, using the notation of Shahani (1979), by

$$F_2 = 1-\frac{(1-f)P_a(n,c)}{P_a(n,c)+\frac{1}{n}\sum_{j=c+1}^{n} jP_j}\ . \tag{56}$$

The AOQ function of the WSP-2 sampling plan, calculated according to (55) with F_2 replacing F_1, has one maximum. Therefore, this plan does not have the unsatisfactory property of the WSP-1 plan, mentioned above. On the other hand, the properties of this plan in the case of quality varying in time are not known.

Shahani (1979) proposed three further modification of the WSP-1 plan, designated as WSP-3, WSP-4, and WSP-5. In the WSP-3 plan the occurrence of the (c+1)st nonconforming item cause the 100% screening of the whole group of N items. For this plan the proportion of inspected is given, according to Shahani (1979), by

$$F_3 = 1 - \frac{nP_a(n,c) + N\{1 - P_a(n,c)\}}{N}. \tag{57}$$

Shahani (1979) proved that $1 - F_1 \geq 1 - F_2 \geq 1 - F_3$. Thus, the WSP-3 plan provides a lower AOQL than the equivalent WSP-1 and WSP-2 plans.

In the WSP-4 and WSP-5 continuous sampling plans another parameter $M=kn$ is added. This is the number of segments that will be diverted to 100% screening as soon as c+1 nonconforming items are found during the sampling phase. According to the WSP-4 plan, if this happens, the next kn segments are screened. In case of the WSP-5 plan, the j sampled segments and the next kn segments undergo 100% inspection. For the WSP-4 plan the proportion of inspected items is given, according to Shahani (1979), by

$$F_4 = 1 - \left[\frac{k\{1 - P_a(n,c)\}}{1 - F_1} - \frac{1}{1 - f}\right]^{-1}, \tag{58}$$

where F_1 is given by (52). The value of F_5 can be found from the relation (Shahani, 1979)

$$\frac{1 - F_1}{1 - F_4} = \frac{1 - F_3}{1 - F_5}. \tag{59}$$

The AOQ function for the WSP-4 plan, depending on the value of k, may be increasing or may have one maximum, but if we choose an appropriate value of k the AOQL value for this plan is always lower than that for the WSP-1 plan. In the case of the WSP-5 plan, the AOQ function has always one maximum, and if we choose an appropriate value of k, we could have the lowest value of the AOQL.

The design of the WSP-type continuous sampling plans requires quite complicated computations. However, Shahani (1979) provided nomograms that could be used for this purpose.

An interesting procedure, similar to the WSP-1 sampling plan, was proposed in the late 1940s in an unpublished presentation by Girshick and later described in (Girshick, 1954). Girshick's sampling plan has three integer parameters m, k and N. The inspected process is divided into segments, each of k items. The inspection begins by inspecting at random one item from consecutive segments. When the cumulative sum of nonconforming items reaches m, the total sample size (i.e. the number of inspected segments) is compared to *N*. If $n \geq N$, the inspected process is considered acceptable. Otherwise, the next *N-n* segments are screened.

For the plan proposed by Girshick (1954) the following inequality holds

$$UAOQL \leq \frac{\left(1-\frac{1}{k}\right)m}{N}. \tag{60}$$

This inequality can be used for choosing the parameters of the plan. Another possibility proposed by Girshick (1954) is to use expressions for the variance of the outgoing quality. Girshick's procedure is the first application of a sequential statistical test in the inspection of continuous processes. It has been extended by Albrecht et al. (1955) who proposed to use Wald's sequential sampling plan for making decisions about switching from sampling to screening.

5. Other approaches to continuous sampling

Continuous sampling plans of CSP-type and WSP-type are not the only sampling procedures that have been proposed for the inspection of continuous processes. There exist continuous sampling plans that retain the original Dodge's idea of switching between screening and sampling, but use other statistical procedures for making decisions for switching. There exist also procedures that do not require explicitly the implementation of 100% screening. In this section we give a short description of those of them which seem to be the most applicable in quality control practice.

It seems to be quite obvious that a good inspection procedure should assure quick switching from screening to sampling when the quality of inspected process is good, and also quick switching from sampling to screening when the inspected process deteriorates. Classical continuous sampling procedures with very simple decision rules do not fulfill this requirement.

It is well known from the theory of mathematical statistics that decision procedures based on sequential tests, such as cumulative sums (CUSUMS), are characterized by shortest inspection runs before making decisions. Therefore, they may be effectively used for the purpose of continuous inspection. Bourke (2002) has proposed such a procedure, designated CSP-CUSUM sampling plan. He proposes to use the same type of data as in the CSP-type procedures, namely run-lengths Y_i, i=1, 2, …, between successive recorded non-conforming items. Decision for switching from screening to sampling is made when the cumulative sum

$$G_i = Max\left[0, G_{i-1} + Y_i - k\right], \quad i = 1,2,\ldots, \quad G_0 = 0 \tag{61}$$

is equal or greater than a prescribed number h_c. Let p_a be an acceptable quality level for which sampling inspection is advisable, and p_r be a rejectable quality level for which 100% screening is needed. From the theory of sequential probability ratio tests (SPRT) one can find the formula for the choice of the parameter *k* to be

$$k = \frac{ln\left[p_r / p_a\right]}{ln\left[(1-p_a)/(1-p_r)\right]} . \tag{62}$$

The Geometric CUSUM scheme defined by (61) provides the effective procedure for switching from screening to sampling when the process quality is or becomes good. A similar procedure can be used for making decisions about switching from sampling to screening. In this case the cumulative sum is calculated as

$$G_i = Max\left[0, G_{i-1} + k - Y_i\right], \quad i = 1,2,\ldots, \quad G_0 = 0 , \tag{63}$$

and a decision is made when this sum is equal to or greater than a prescribed number h_s. The parameter k is the same for both CUSUMs.

Statistical characteristics of the CSP-CUSUM procedure have been investigated by Bourke (2002) using the Markov chain approach. He compared the CSP-CUSUM procedure with other CSP-type plans (CSP-1, CSP-2) taken from the MIL-STD-1235 standard. For making comparisons Bourke (2002) introduced a new measure of performance, namely the Average Cycle Length (ACL) defined as

$$ACL(p) = ANI100(p) + n\left[ANISAM(p)\right], \tag{64}$$

where $n=1/f$, $ANI100(p)$ is the average length of the screening phase, and the $ANISAM(p)$ is the average number of samples inspected during the sampling phase. The value of ACL shows us how often the inspection process returns to its screening phase, and its desirable values should be sufficiently high. Bourke (2002) compared CSP-1, CSP-2, and CSP-CUSUM sampling plans that have been characterized by the same AOQL value, and showed on examples that the minimal ACL value for equivalent CSP-CUSUM procedures is much higher than for respective CSP-1 and CSP-2 plans. In his paper Bourke (2002) presents a table of suggested CSP-CUSUM continuous sampling plans indexed by preferred AQLs (or the values of the AOQL).

Another continuous sampling plan based on run-lengths was proposed in (Bourke, 2003), and designated CSP-SUM. This plan is based on the following statistic

$$RL_2 = Y_{i-1} + Y_i, \quad i = 1,2,\ldots, Y_0 = 0 , \tag{65}$$

where Y_i is a number of consecutive conforming items between the $(i\text{-}1)^{th}$ and i^{th} nonconforming items found during inspection. Inspection is switched from screening to sampling when the observed value of RL_2 exceeds a critical value U. When sampling begins, calculation of RL_2 is restarted. The sampling inspection switches back to 100% screening when the value of RL_2 falls below another critical value L. Bourke (2003) presents a table of suggested CSP-SUM continuous sampling plans indexed by preferred AQLs (or the values of the AOQL). He shows on examples that the performance of the CSP-SUM is only slightly worse than the performance of the CSP-CUSUM, and significantly better than the performance of equivalent CSP-1 and

CSP-2 plans. Taking into account its simplicity, the CSP-SUM should be regarded as a valuable option for the inspection of continuous processes.

An original, and easily implemented, continuous sampling procedure was proposed by Beattie (1962). Beattie's procedure is the application of two CUSUM control charts. Samples of n items each are taken from an inspected process, and the numbers of nonconforming items, d_i, i=1,2,…are used for the calculation of cumulative sums

$$S_i = Max\left[0, S_{i-1} + d_i - k\right], \quad i = 1,2,\ldots, \quad S_0 = 0\,, \tag{66}$$

where k is a reference value proportional to the slope of Wald's sequential sampling plan. Charting is continued in the accept zone as long as $S_i<h$, where h is a parameter of the chart. If $S_i \geq h$, the procedure is switched to the reject zone, and the CUSUM plot is moved up to the value $h+h^*$. The charting is continued in the reject zone, using cumulative sums

$$S_i = Min\left\lfloor h + h^*, S_{i-1} + d_i - k\right\rfloor, \tag{67}$$

until $S_i \leq h$. When $S_i \leq h$, the CUSUM plot is restarted at 0. In his paper Beattie (1962) proposes to reject (i.e. to throw away, or put on side for 100% inspection if desired) all produced items while the inspection process remains in the reject zone.

For the calculation of the characteristics of his procedure, Beattie (1962) used a general methodology proposed for CUSUM charts by Ewan and Kemp (1960). According to this methodology, two systems of linear equations have to be solved. In the case of the CUSUM procedure in the accept zone we have to solve, with respect to the function $L(z,p)$ describing the expected number of inspected samples, the following equations

$$L(z,p) = 1 + \sum_{y=1}^{h-1} L(y,p)\varphi(y+k-z) + L(0,p)\sum_{x=0}^{k-z}\varphi(x), \quad z = 0,1,\ldots,h-1\,. \tag{68}$$

In the case of the reject zone, a similar system is given by the following formula

$$L^*(z,p) = 1 + \sum_{y=1}^{h^*-1} L^*(y,p)\varphi(y+k-z) + L^*(0,p)\sum_{x=z+k}^{\infty}\varphi(x), z = 0,1,\ldots,h^*-1\,. \tag{69}$$

In both (68) and (69) $\varphi(x)$ is the probability function, binomial or Poisson, that depends on the fraction nonconforming p, and determines the probability of observing in the sample of r items exactly x nonconforming items. The Average Run Length in the accept zone ARL_0, i.e. the expected number of inspected samples until the process switches to the reject zone, is given by $L(0,p)$. The same characteristic in the reject zone, ARL_1, is given by $L^*(0,p)$. When the sampling rate is the same in both zones, the OC curve, understood as the proportion, P_A, of product accepted, for a given quality p, is given by

$$P_A(p) = \frac{L(0,p)}{L(0,p) + L^*(0,p)}. \tag{70}$$

The graphs of ARLs in both zones were presented by Prairie and Zimmer (1973) for different combinations of n, k, h and h^*. These graphs can be used for the determination of these parameters, i.e. for the design of Beattie's procedure.

It is worth noting that Beattie (1962) leaves the problem of the frequency of sampling open. Therefore, his procedure cannot be directly compared to other continuous sampling procedures. Zimmer and Tai (1980) considered the case when the sampling rate (i.e. the proportion of sampled items) in the acceptance zone is equal to r_a, and the sampling rate in the reject zone is equal to r_r. If $r_r<1$ (i.e. less than 100% items are inspected in that zone), and $r_r>r_a$, the average fraction of total product inspected in the long run is (Zimmer and Tai, 1980)

$$F(p) = \frac{L(0,p) + L^*(0,p)}{\frac{1}{r_a}L(0,p) + \frac{1}{r_r}L^*(0,p)}. \tag{71}$$

Hence, the AOQ function is given by

$$AOQ = (1 - r_r)p + (r_r - r_a)pP_A(p). \tag{71}$$

where $f=r_a/r_r$, and the probability of acceptance $P_a(p)$ is given by

$$P_A(p) = \frac{L(0,p)}{L(0,p) + fL^*(0,p)}. \tag{72}$$

When $r_r=1$, or when $r_r<1$ but the rejected material is diverted for 100% screening, the AOQ function is given by

$$AOQ = (1 - r_a)pP_A(p). \tag{73}$$

In both cases considered above, $P_a(p)$ is computed with $r_r=1$.

Zimmer and Tai (1980) showed that slightly modified Dodge's CSP-1 and CSP-2 continuous sampling plans are special cases of the Beattie procedure. Suppose that the Beattie procedure begins in the reject zone, i.e. $S_0=h+h^*$. If we take $n=1$, $r_r=1$, $r_a=f$, $k=1/i$, $h=1-1/i$, and $h^*=1$, the Beattie procedure is the same as the CSP-1 sampling plan. Moreover, when we set $n=1$, $r_r=1$, $r_a=f$, $k=1/(l+1)$, $h=1$, and $h^*=i/(l+1)$, where l is the release parameter in the sampling phase of the CSP-2 sampling plan, the Beattie procedure is the same as the CSP-2 sampling plan. Thus, CSP-1 and CSP-2 are special cases of Beattie's procedure for sampling by attributes.

Beattie's procedure is a combination of two CUSUM procedures. Thus, it can be used not only for sampling by attributes, but – as was already noticed by Beattie (1962) – also for sampling by variables. Wasserman and Wadsworth in their papers, Wadsworth and Wasserman (1987) and Wasserman and Wadsworth (1989) considered a general case when monitored quality characteristic is distributed according to the probability distribution that belongs to the general family of the exponential distributions (Darmois-Koopman family) defined by the pdf function

$$f(x;\theta)=C(\theta)a(x)exp[D(\theta)T(x)], \tag{74}$$

where $C(\theta)>0$, $a(x)>0$, and $D(\theta)$ is a strictly increasing function of θ. It is well known that the popular SQC probability distributions such as the binomial, Poisson and normal belong to this family.

Statistical properties of Beattie's procedure can be analyzed using general results presented in Wald (1947). Results of this analysis for the general case are given in Wadsworth and Wasserman (1987). Let θ_0 represent the good process quality level, and θ_1 represent the bad process quality level. The SPRT reference parameter *k* (the reference parameter of the CUSUM procedure) according to Wald's theory should be computed from the following formula

$$k=\frac{ln[C(\theta_0)/C(\theta_1)]}{\Delta D}, \tag{75}$$

where $\Delta D=D(\theta_1)-D(\theta_0)$. Let $g(\theta)$ be the function satisfying the famous Wald's identity

$$E_X[g(\theta)S_N \mid \theta]=1, \tag{76}$$

where S_N is the SPRT statistic upon termination of the test. Wasserman has shown, see Wadsworth and Wasserman (1987) for detailed information, that the Average Run Length of Beattie's procedure in the accept zone is given by

$$L(\theta)=\frac{h}{\theta-k}\left[1+\frac{exp(-\eta)-1}{\eta}\right], \tag{77}$$

where $\eta=-h\Delta Dg(\theta)$. Similarly, in the reject zone the ARL function is given by

$$L^*(\theta)=\frac{h^*}{\theta-k}\left[-1+\frac{exp\left(-\eta^*\right)-1}{\eta^*}\right], \tag{78}$$

where $\eta=-h^*\Delta Dg(\theta)$. In the derivation of (77) and (78) it is assumed that the reference value *k* is the same for both zones.

Read and Beattie (1961) introduced the Type C OC curve for continuous acceptance sampling as the long run proportion of product that is accepted. The approximate expression for this characteristic was given in Wadsworth and Wasserman (1987)

$$P_a(\theta) \approx \frac{\eta - 1 + exp(-\eta)}{\eta - \eta^* - 2 + exp(-\eta) + exp(\eta^*)}, \quad \eta, \eta^* \neq 0\,. \tag{79}$$

This characteristic has been used in Wadsworth and Wasserman (1987) for designing Beattie's procedure.

Wasserman, see Wadsworth and Wasserman (1987), has shown that Beattie's procedure with $k=k^*$ and $h=h^*$ (i.e. with $\eta = \eta^*$) gives the best "discrimination" between processes described by θ_0 and θ_1, respectively. This discrimination is measured by the difference $P_a(\theta_0) - P_a(\theta_1)$. Thus, Wadsworth and Wasserman (1987) used this assumption for the construction of the procedure. They also assumed that for the process of "good" quality, characterized by the parameter θ_0, the fraction of accepted product should be greater than 1-α. On the other hand, for the process of "bad" quality, characterized by the parameter θ_1, the fraction of accepted product should be smaller than β. Moreover, they assumed that the average number of samples inspected in the reject zone, if the process is "good", should be smaller than L^0. On the other hand, the average number of samples inspected in the accept zone, if the process is "bad", should be smaller than L^1. In case of sampling by attributes (described by the Poisson distribution) the algorithm proposed in Wadsworth and Wasserman (1987) is the following:

1) Specify L^0, L^1, θ_0, θ_1, α, β.
2) Calculate k from
$$k = \frac{\theta_1 - \theta_0}{ln\left(\theta_1 / \theta_0\right)}$$
3) Find η_0 from $P_a(\eta_0) \geq 1 - \alpha$, and η_1 from $P_a(\eta_1) \leq \beta$.
4) Let $\eta_{max}=\max\{\eta_1, -\eta_0\}$.
5) Calculate h from
$$h = \frac{\eta_{max}}{ln\left(\theta_1 / \theta_2\right)}\,.$$
6) Calculate the sample size n as the smallest integer not smaller than
$$n^* = \frac{2h}{\theta_1 - \theta_0} max\left\{\frac{1}{L_0}, \frac{1}{L_1}\right\}.$$

The detailed description of this procedure in the case of the normal distribution is given in Wasserman and Wadsworth (1989) for $\theta_0=\mu_0$, $\theta_1=\mu_1$, known σ^2, and $L^1=L^0$.

1) Specify L^0, μ_0, μ_1, σ^2, α, β.
2) Let k=0,5(μ_0+μ_1).
3) Find η_0 from $P_a(\eta_0) \geq 1-\alpha$, and η_1 from $P_a(\eta_1) \leq \beta$.
4) Let $\eta_{\max}=\max\{\eta_1, -\eta_0\}$.
5) Let
$$h = \frac{\mu_1 - \mu_0}{2} L^0 .$$
6) For item-by-item sampling, verify that
$$\left(\frac{\mu_1 - \mu_0}{\sigma}\right)^2 \geq 2\frac{\eta_{max}}{L^0} ;$$
otherwise, choose n as the nearest integer to quantity
$$\frac{\sigma^2 \eta_{max}}{h(\mu_1 - \mu_0)} .$$

6. Economically optimal procedures for monitoring continuous processes

Every implementation of sampling inspection has some economic consequences. In the case of sampling of continuous processes one can distinguish two general cases: "troubleshooting" (Chiu and Wetherill, 1973), and product screening. The aim of product screening is simply to rectify the process. The "troubleshooting" is used to determine if the process has deteriorated. It is usually assumed that a continuous process may be either in a "good" state (state 1) or in a "bad" state (state 2). The transition from the "good" state to a "bad" one is not directly observed, and can be revealed by an appropriate sampling procedure. When the process is judged to be in a "bad" state it has to be stopped, and the search for an assignable cause of this situation should begin. It is also assumed that a deteriorated process cannot be improved without some repair actions.

The first attempt to use an economic approach in the design of a continuous sampling procedure was proposed in the paper by Girshick and Rubin (1952). They assumed that after each item produced in state 1 there is a constant probability, p, that the process will jump to state 2. It is assumed that the quality characteristic X has the probability distribution $f_1(x)$ when the process is in state 1, and the probability distribution $f_2(x)$ when the process is in state 2. Moreover, it is assumed that the value of the item of quality x is $V(x)$. When the process is stopped in state i (i=1,2), it takes n_i time units (a time unit is taken to be the time of the production of one item) for inspection and repairing, and the costs of these actions per time unit are equal to c_i (i=1, 2).

Girshick and Rubin (1952) considered two cases. In the first one they assumed that the cost of sampling is negligible, and thus the 100% inspection is used. They applied a Bayesian approach and showed that the optimum stopping rule is to stop the process if after the inspection of the k^{th} item the condition $Z_k \geq a$, where

$$Z_0 = 0, Z_k = y_k\left(1 + Z_{k-1}\right), \tag{80}$$

and

$$y_k = \frac{f_2\left(x_k\right)}{\left(1-p\right)f_1\left(x_k\right)}, \tag{81}$$

is fulfilled. The critical value a can be found by solving an integral equation and then maximizing the average income.

In the second case that they considered, Girshick and Rubin (1952) assumed that the inspection process is costly. In this case the optimum rule is also defined in terms of Z_k calculated according to (80), but the value y_k is now calculated as

$$y_k = \frac{1}{1-p}. \tag{82}$$

The decision rule is now the following: produce the next item without inspection if $Z_k<b$, inspect the next produced item if $b \leq Z_k < a$, and stop the process if $Z_k \geq a$. The critical values *a,* and *b* are calculated in a similar way as in the first case mentioned above. The results of Girshick and Rubin (1952) have been generalized by many other authors. Some information about those results can be found in Chiu and Wetherill (1973).

When a process is characterized by a constant probability of producing a nonconforming item, and functions describing costs of inspection, repair or replacement, and costs of passing a nonconforming item are linear, then the economically optimal procedure is either to inspect all items (100% screening) or to do nothing. The first option is optimal when $p \geq p_B$, and the second option is optimal when $p \leq p_B$, where p_B is a certain quality level, called "break-even quality". This result was firstly proved for the inspection of lots, see Hald (1981) for more information, and later for the inspection of continuous processes. Vander Wiel and Vardeman (1994) have shown that this result is valid for a more general model of process behavior. Therefore, sampling inspection of continuous processes may be optimal only in the case of quality p that varies in time around the value of p_B.

Anscombe (1958) was the first author who used a cost model for finding an optimal procedure of the CSP type. Let s be the cost of passing a nonconforming item, r be the cost of repair or replacement of a nonconforming item found during the inspection, and c be the unit cost of inspection, and

$$k = p_B = \frac{c}{s-r} \tag{83}$$

be the "break-even quality".

Anscombe (1958) made the following additional assumptions:

a) On the average, the process deteriorates after M items have been produced;
b) Process quality p varies randomly according to a uniform distribution in the range $(0, 2k)$;
c) The fraction of items inspected at k is equal to 0.5, i.e. $F(k)=0.5$.

Using these assumptions he found that the optimal parameters of the CSP-1 plan can be calculated from the formulae

$$f = (0{,}3kM)^{-\frac{2}{3}}, \tag{83}$$

and

$$i = \frac{ln\left(\frac{f}{1-f}\right)}{ln(1-p_B)} \approx \frac{1}{p_B} ln\left(\frac{1}{f}\right). \tag{84}$$

The results obtained by Anscombe (1958) can be justified using the mini-max approach to optimization of sampling procedures. Let $V(p)$ be the expected cost per unit produced that includes costs of inspection, costs of repair or replacement of nonconforming items found during inspection, and costs (losses) due to passing nonconforming items. The costs described by $V(p)$ consist of two parts, and one of them $U(p)$ describes costs that are unavoidable for any inspection procedure. The difference

$$R(p) = V(p) - U(p) \tag{85}$$

is called the regret function, and describes additional costs related to the applied inspection procedure. The mini-max approach for the design of a sampling procedure consists in finding such parameters of this procedure for which the function

$$R^*(p) = \max_{0 \le p \le 1} R(p) \tag{86}$$

is minimized. Thus, a mini-max optimal procedure guarantees the lowest costs in the most unfavorable situation.

The first paper devoted to the problem of the mini-max optimization of the CSP-1 continuous sampling plan was published by Ludwig (1974). He proposed the following cost function

$$V(p) = c + (a + bp)F(p), \tag{87}$$

where c is the average cost of producing an item (the average profit from a produced item, if c is negative), a is the cost of inspection of one item, b is the cost of repair or replacement of a nonconforming item found during the inspection, and $F(p)$ is the fraction of inspected items. Moreover, Ludwig (1974) assumed that the sampling procedure should assure that the average outgoing quality should not be worse than L_1. Then, he defined the regret function as

$$R(p) = \begin{cases} (a+bp)F(p) & 0 \le p \le L_1 \\ (a+bp)[F(p)-1+L_1/p] & L_1 < p \le 1 \end{cases} \tag{88}$$

The mini-max optimal plans for this regret function have been tabulated in Ludwig (1974) for different combinations of L_1 and $d=a/b$.

Another mini-max model for the optimization of the CSP-1 plan has been proposed by Vogt (1986). He assumed a typical cost model

$$V(p) = (a+bp-ps)F(p) + ps\,, \tag{89}$$

where costs a and b are the same as in the model by Ludwig (1974), and s is the cost of passing a nonconforming item. The standardized regret function proposed by Vogt (1986) is described by the following formula

$$R(p) = \begin{cases} (p_B - p)F(p) & 0 \le p \le p_B \\ (p_B - p)[1-F(p)] & p_B < p \le 1 \end{cases}. \tag{90}$$

For this regret function Vogt (1986) proposed an algorithm for finding approximately mini-max optimal parameters of the CSP-1 plan.

In the classical approach to the economic optimization of sampling procedures used in lot-by-lot inspection it is assumed that quality varies randomly from lot to lot. This approach can be applied in the case of plans of the CSP-type if we assume that the inspected process is divided into many segments of finite and long length, and the quality varies randomly from segment to segment. This approach is described in the paper by Hryniewicz (1988). The optimal results are characterized by large values of the release parameter i, and the very small values of the sampling rate f. This result is hardly unexpected. The role of the first screening period is to verify if the quality is better than the “break-even quality”. If it is so, the further inspection is not necessary. Otherwise, the whole segment should be screened.

7. Continuous sampling plans in standards

The first standards on continuous sampling plans were published in the US Department of Defense in the 1950s (H-106, H-107). Another standard of that type (QSTAG 340) was developed together for the Armies of the United States, the United Kingdom, Canada and Australia in the 1970s. The most popular standard, developed as the successor to H-106 and H-107, was published under designation MIL-STD-1235A in 1974. Its latest version, MIL-STD-1235C (1988), is now available on the Web free of charge.

MIL-STD-1235C (1988) provides tables, figures, and procedures for five types of continuous sampling plans by attributes:

- CSP-1 continuous sampling plans;
- CSP-F continuous sampling plans for finite production runs;
- CSP-T three-level continuous sampling plans ;
- CSP-2 continuous sampling plans;
- CSP-V continuous sampling plans (similar to CSP-2 and CSP-3 sampling plans).

Sampling plans in MIL-STD-1235C are indexed by Sampling Frequency Code Letter that determines parameter f of the plan, and Acceptable Quality Level (AQL) which for continuous sampling plans is simply an index, and does not have any precisely defined meaning.

Recommended sampling frequencies form a series of preferred frequencies from 1/2 (code letter A), to 1/200 (code letter K). The choice of an appropriate code letter depends upon the number of units in a production interval that shall not be longer than one day's production.

Clearance numbers, i, are chosen from tables for given pairs of code letter and AQL. In sampling plans of MIL-STD-1235C the length of the screening phase is limited by a constant S, i.e. when the number of items inspected during the screening phase reaches S, inspection is suspended and corrective actions shall take place.

The operation procedures of CSP-1 and CSP-2 plans are the same as has been presented in sections 2.1 and 2.2, respectively. The three-level CSP-T sampling plan operates similarly as the original CSP-T plan described in section 3, but the sampling frequencies on the consecutive levels are f, $f/2$, and $f/4$, instead of f, f^2, and f^3. The operation procedure of the CSP-F sampling plan is exactly the same as in the case of the CSP-1 sampling plan. However, its parameters have been calculated under the assumption of the finite length of production runs. CSP-F sampling plans are organized in tables indexed by AQL and AOQL values. Possible finite production runs have been divided into 32 non-overlapping intervals. In a given table for each of these intervals eight sampling plans (i,f) are available. CSP-V sampling plan operates like the CSP-2 sampling plan. However, when a second nonconforming item is found during the sampling phase, additional 100% inspection of consecutive k items is invoked (as in the CSP-3 plan). The procedure is switched to the screening phase when a nonconforming item is found during this 100% inspection. Otherwise, the sampling is continued as in the CSP-2 plan.

MIL-STD-1235C has over 300 pages, and on over 200 of them different curves describing statistical characteristics of proposed plans (except for CSP-F plans), such as e.g. AOQ curves, are given.

8. Future work on new standards for continuous sampling plans

Statistical procedures proposed in standards should fulfill certain important requirements. First of all, they should be based on a firm mathematical basis.

Secondly, they should be easily implemented in practice. And finally, they should use similar language to that used by practitioners who work with their predecessors or other similar standards. Future new standards on acceptance sampling of continuous processes should definitely fulfill these requirements.

Classical standardized acceptance sampling plans are used at an interface between a "producer" and a "user". Therefore, the amount of information that is needed for their design and further implementation should be limited to that available by both "partners". This seems to be very restrictive in many practical cases, and – for example – limits the possibility of using economic considerations in construction of sampling procedures. In contrast to that, continuous sampling may be mainly used inside the same organization which is able to collect information of a confidential character. Therefore, in future standardized procedure we should use the results presented in section 6 showing that in case of stable processes the economically optimal behaviour of the process owner is either to screen 100% of produced items or to do nothing. Hence, the main aim for using such procedure is to confirm that the inspected process is in a "good" state, and to indicate as quickly as possible that it has just deteriorated and needs repair. In order to do so we need to introduce in a standard the concept of the "break-even quality". To do so, we propose to use the concept of *interval evaluation* of unit costs.

Let (c_{min},c_{max}) be the interval of possible values of the unit cost of inspection, (r_{min},r_{max}) be the interval of possible values of the unit cost of repair or replacement of nonconforming items found during the inspection, and (s_{min},s_{max}) be the interval of possible values of the cost incurred by nonconforming items that have been not detected by the inspection procedure. Thus, we can easily find the interval representation $(p_{B,min},p_{B,max})$ of the "break-even quality" defined e.g. by (83), where

$$p_{B,min} = \frac{c_{min}}{s_{max} - r_{min}} \tag{91}$$

and

$$p_{B,max} = \frac{c_{max}}{s_{min} - r_{max}}, \; s_{min} > r_{max}. \tag{92}$$

In the design of the sampling procedure we have to take into account that, for a process quality (even varying in time) better than $p_{B,min}$, any sampling activities create unnecessary costs. Therefore, when the inspected process remains in a "good" state the inspection procedure in the sampling phase – which usually takes place for processes of "good" quality - should ensure:

a) a small proportion of inspected items,

b) a small probability of erroneous switching from sampling to screening (small probability of a false alarm),

c) a large probability of correct switching from sampling to screening when the process quality becomes worse than $p_{B,max}$.

Unfortunately, these requirements are in conflict. Requirements b) and c) mean strong discrimination between qualities better than $p_{B,min}$ and worse than $p_{B,max}$. This cannot be achieved without taking large samples, and this stands in conflict with the requirement a). We face an only slightly different situation when the inspection process operates in its screening phase. The sampling procedure should in this case ensure a quick return to the sampling phase when the quality is or becomes "good". On the other hand, however, when the process is "bad", the probability of switching from screening to sampling should be small. Thus, in this case we also face the problem of good discrimination between two close quality levels. The only difference, when compared to the first case considered, is related to the fact that in the "bad" state we are not afraid of inspecting large amount of items. In fact in this state, 100% screening is the optimal inspection policy. However, we must take into account that a process that remains in a "bad" state is completely unacceptable. Therefore, the permanent phase of screening cannot be accepted, and when this happens, the process should be stopped and the assignable cause of this situation should be found and removed.

The requirements presented above show undoubtedly that the inspection process should start, as in the Beattie procedure, from the sampling phase. In order to have good discrimination with the lowest possible sampling effort we have to use sequential sampling tests. Therefore, potential candidates for this purpose are *Beattie's sampling procedure*, i.e. an ordinary CUSUM, or *Bourke's CSP-CUSUM* continuous sampling plan, i.e. a geometric CUSUM for runs. When the inspection process enters the screening phase, the optimal procedure is *Wald's curtailed sequential sampling plan*. In this case the decision of acceptance shall cause the return to the sampling phase, and the decision of rejection should be equivalent to the decision of either to stop the process or to continue the 100% screening.

Parameters of sampling plans can be calculated independently for both phases of inspection. This gives additional degrees of freedom for the designer of the procedure who can use some additional criteria for choosing the best one. The concept of the ACL introduced by Bourke (2002) and defined by (64) can be used for this purpose. It seems that maximization of

$$ACL_{min} = \min_{p \le p_{B,min}} ACL(p) \tag{93}$$

could yield the procedure, which for "good" processes operates mainly in the sampling phase. Another feature worthy considering is the discrimination level of the procedure in question. The highest discrimination may be achieved if we require low probabilities of false decisions for process qualities $p_{B,min}$ and $p_{B,max}$. One can decrease the discrimination rate, and therefore decrease the inspection effort, by replacing $p_{B,min}$ with the average quality $\overline{q}$ of the process being in a "good" state. The simplest possibility, however, is to use the results of Wasserman and Wadsworth (1989) for Beattie's procedure taking into account different sampling rates in both phases of the inspection process.

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